Permutation Representations With Application To Quasicrystals And Carbon Nanotubes

Nicolae Cotfas

Faculty of Physics, University of Bucharest, Romania, ncotfas@yahoo.com

Let $\{g:\mathbb{R}^n\longrightarrow\mathbb{R}^n\ |\ g\in G\}$ be an orthogonal $\mathbb{R}$-irreducible representation of a finite group G, and let $\mathscr{C}=\{e_1,...,e_k,-e_1,...,-e_k\}$, where $e_i=(e_{i1},e_{i2},...,e_{in})$, be a finite union of orbits of G symmetric with respect to the origin. For each $g\in G$, there exist $s_1^g,\ s_2^g,...,s_k^g\in\{-1;\ 1\}$ and a permutation of the set $\{1,2,...,k\}$ denoted also by g such that $ge_j=s_{g(j)}^g e_{g(j)}$ for any $j\in\{1,2,...,k\}$. The formula $g\varepsilon_j=s_{g(j)}^g\varepsilon_{g(j)}$, where $\{\varepsilon_1,\varepsilon_2,...,\varepsilon_k\}$ is the canonical basis of $\mathbb{R}^k$, defines the *permutation representation* $g(x_1,x_2,...,x_k)=(s_1^g x_{g^{-1}(1)},s_2^g x_{g^{-1}(2)},...,s_k^g x_{g^{-1}(k)})$ of G in $\mathbb{R}^k$.

The subspace $\mathbf{E}=\{\ (<u,e_1>,...,<u,e_k>)\ |\ u\in\mathbb{E}_n\ \}$ of $\mathbb{R}^k$ is G-invariant, and in view of Schur's lemma the vectors $w_1=\kappa^{-1}(e_{11},...,e_{k1}),\ ...,\ w_n=\kappa^{-1}(e_{1n},...,e_{kn})$ where $\kappa=\sqrt{(e_{11})^2+(e_{21})^2+...+(e_{k1})^2}$, form an orthonormal basis of $\mathbf{E}$. The subduced representation of G in $\mathbf{E}$ is equivalent with the representation of G in $\mathbb{R}^n$, and the isomorphism of representations $\mathscr{I}:\mathbb{R}^n\longrightarrow\mathbf{E}$, $\mathscr{I}(\alpha_1,\alpha_2,...,\alpha_n)=\alpha_1w_1+...+\alpha_nw_n$ allows us to identify the 'physical' space $\mathbb{R}^n$ with the subspace $\mathbf{E}$ of the *superspace* $\mathbb{R}^k$.

By embedding the space $\mathbb{R}^n$ into the higher-dimensional space $\mathbb{R}^k$, the intial representation of G is replaced by an equivalent representation such that the group transformations are signed permutations. In the new representation the mathematical expressions of the G-invariant objects are simpler and more symmetric. The description of $\mathbb{R}^n$ in terms of the superspace $\mathbb{R}^k$ is similar to a description in terms of coherent states used in quantum mechanics. The construction presented above can be regarded as a version for finite groups of the coherent states defined by Perelomov in the case of Lie groups.

The use of a superspace offers some mathematical facilities, very important in the description of physical systems. For example, the orthogonal projection $\mathscr{Q}=\pi(\mathbb{Z}^k\cap\mathscr{S})$ on the physical space $\mathbf{E}$ of the set of all the points of $\mathbb{Z}^k$ lying inside the strip $\mathscr{S}=\mathbf{E}+[0;1]^k$ generated by shifting along $\mathbf{E}$ the unit hypercube $[0;1]^k$ plays an important role in the description of the atomic structure of quasicrystals. The superspace $\mathbb{R}^k$ offers some important facilities in the study of the self-similarities of $\mathscr{Q}$ and in the description of phasons in quasicrystals.

The three-axis description of the honeycomb lattice is directly related to the use of a three-dimensional superspace. There is a natural bijection between the set of all the vertices of a honeycomb lattice and the set $\mathscr{L}=\{(x_1,x_2,x_3)\in\mathbb{Z}^3\ |\ x_1+x_2+x_3\in\{0;1\}\}$ which is an alternate mathematical model, and the starting point for a new mathematical approach to carbon nanotubes. More details are available online at `http://fpcm5.fizica.unibuc.ro/~ncotfas/`.